Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

If required by your examination administrator, please place your student identification label in this box

Section Two: Calculator-assumed

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	100	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Markers use only		
Question	Maximum	Mark
8	5	
9	6	
10	8	
11	11	
12	11	
13	8	
14	9	
15	10	
16	6	
17	9	
18	10	
19	7	
S2 Total	100	
S2 Wt (×0.65)	65%	

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Section Two: Calculator-assumed **65% (100 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(5 marks)**

A hire company have a fleet of n bicycles in a city. On any given day, the probability that one of their bicycles needs a repair is independent with a constant value of p.

The random variable X is the daily number of bicycles needing a repair and it has a mean of 53.76 and standard deviation 6.72.

(a) Determine the value of n and the value of p. (3 marks)

(b) The daily cost to the hire company of these repairs C, in dollars, is also a random variable. It consists of a fixed amount of $\$840$ to cover workshop and labour costs plus an average of $\$38.50$ per bicycle repaired for parts and consumables.

Determine the mean and standard deviation of the daily repair cost. (2 marks)

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Question 9 **(6 marks)**

A barrel is filled with 34 balls numbered with the integers $1, 2, 3, \dots, 33, 34$, but otherwise identical.

Let the random variable X be the number on a ball drawn at random from the barrel.

(a) Explain why X has a uniform distribution. (1 mark)

(b) Determine the expected value of X. (1 mark)

Let the random variable Y take the value 1 when $X < 10$ and the value 0 otherwise.

(c) State the particular name given to two-outcome random variables such as Y. (1 mark)

(d) Determine $\mathrm{P}(Y = 1)$. (1 mark)

(e) Three balls are drawn at random from the barrel with each being replaced before the next is taken. Determine the probability that exactly two of the balls are marked with single digit numbers. (2 marks)

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Question 10 **(8 marks)**

45 mg of a radioisotope with a half-life of 77 hours was injected into a patient before a CT scan. The mass M of the radioisotope decays continuously so that t hours after administration, the mass remaining is given by $M = M_0 e^{-kt}$, where M_0 and k are constants.

(a) Determine the value of the constants M_0 and k. (3 marks)

(b) Determine the mass of the radioisotope that remains in the patient exactly one week after their injection. (1 mark)

(c) Eventually, the mass of the remaining radioisotope falls to 5 mg.

(i) Determine how long after their injection that this occurs. (2 marks)

(ii) Determine the rate at which the radioisotope is decaying at this time. (2 marks)

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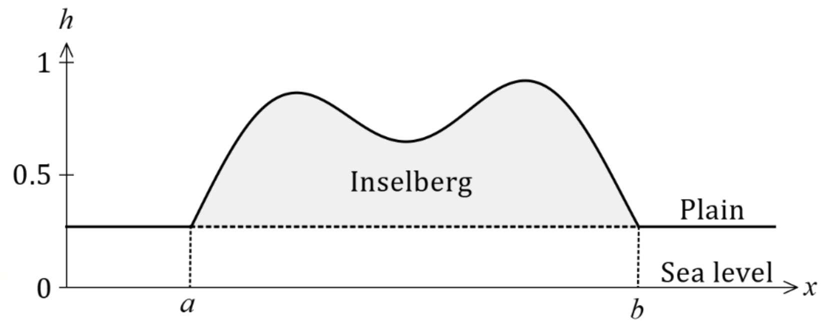
Question 11 **(11 marks)**

A vertical cross section through the highest point of an inselberg, a mountain range that rises above a surrounding level plain, is shown in the figure below.



The height of the plain and the inselberg above sea level h, in kilometres, is given by

$$h(x) = \begin{cases} x - \dfrac{1}{5}\left(x^2 + 2 + \sin\left(\dfrac{13x}{4}\right)\right) & a \le x \le b \\ 0.27 & \text{otherwise} \end{cases}$$

where x is the horizontal displacement in kilometres from an arbitrary origin.

(a) Determine the value of a and the value of b, the x displacements where the inselberg meets the surrounding plain. (2 marks)

(b) Use calculus to determine the cross-sectional area of the inselberg shaded in the figure above. (3 marks)

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(c) Use calculus to

(i) determine the maximum height of the inselberg above the surrounding plain. (4 marks)

(ii) verify that the stationary point on the curve that represents the highest part of the inselberg is a maximum. (2 marks)

Question 12 **(11 marks)**

A random sample of 150 households within a large town revealed that 48 households owned a cat, 60 owned a dog and 27 owned both a cat and a dog. You may assume that point estimates of probabilities derived from this sample are reliable and representative of the whole town.

(a) For households within the town, determine the probability that

(i) a randomly selected household owns neither a cat nor a dog. (2 marks)

(ii) in a random sample of 5 households, exactly 3 will not own a dog. (3 marks)

(iii) in a random sample of 9 households that own a dog, at least 2 will own a cat. (3 marks)

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(b) If another random sample of 276 households was drawn from within the town, determine the mean and standard deviation of the probability distribution that models the number of households that own either a cat or a dog in the sample. (3 marks)

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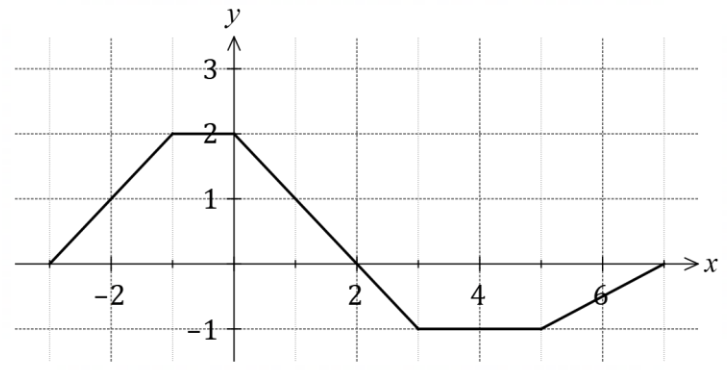
Question 13 **(8 marks)**

The graph of $y = f(x)$ is shown below.



Evaluate each of the following.

(a) $\int_{-2}^{4} f(x)\,dx.$ (2 marks)

(b) $\int_{0}^{7} (f(x) + 2)\,dx.$ (2 marks)

(c) $\int_{-1}^{-3} 3f(x)\,dx.$ (2 marks)

(d) $\int_{0}^{5} f'(x)\,dx.$ (2 marks)

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Question 14 **(9 marks)**

A particle is moving in a straight line with acceleration $a = 3e^{-0.25t}$ cm/s^2 after t seconds. When $t = 0$ it has a displacement of 1.5 m and a velocity of -7 cm/s.

(a) Determine the acceleration of the particle at the instant at which it comes to rest. (4 marks)

(b) Determine an expression for the displacement of the particle in terms of t. (2 marks)

(c) Determine the velocity of the particle when it again has a displacement of 1.5 m. (3 marks)

Question 15 **(10 marks)**

(a) Use the quotient rule to show that $\frac{d}{dx}\left(\frac{4x+2}{e^{0.5x}}\right)=\frac{3}{e^{0.5x}}-\frac{2x}{e^{0.5x}}$. (3 marks)

(b) Use your result from part (a) to show that $\int \frac{2x}{e^{0.5x}}\,dx=\frac{-4x}{e^{0.5x}}-\frac{8}{e^{0.5x}}+c$, where c is a constant. (3 marks)

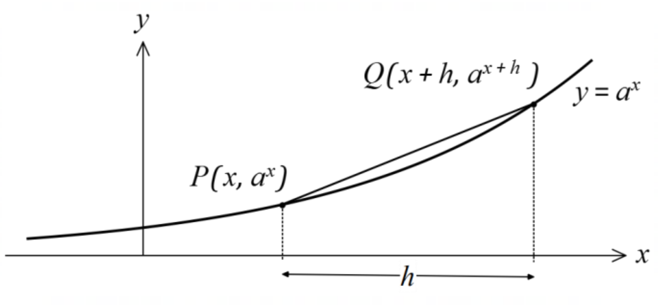
(c) The height h of a plant, initially 9 cm, is changing at a rate given by $\frac{dh}{dt} = \frac{2t}{e^{0.5t}}$ cm/day, for $t \geq 0$.

(i) Determine an equation to model the height of the plant as a function of time and hence determine its height after 7 days. (3 marks)

(ii) According to the model, what height will the plant never exceed? (1 mark)

Question 16 **(6 marks)**

The graph of $y = a^x$ is shown in the diagram below, where a is a positive constant.



A secant is drawn between points P and Q that lie on the curve with x-coordinates x and $x + h$ respectively.

(a) Describe the property of the secant that $\dfrac{a^{x+h} - a^x}{h}$ represents. (1 mark)

(b) Describe the property of the curve that $\lim\limits_{h\to 0}\left(\dfrac{a^{x+h} - a^x}{h}\right)$ represents. (1 mark)

It can be shown that $\lim\limits_{h\to 0}\left(\dfrac{a^{x+h} - a^x}{h}\right) = a^x \lim\limits_{h\to 0}\left(\dfrac{a^h - 1}{h}\right)$.

(c) Complete the following table when $a = 3$, rounding values to 4 decimal places, and explain how the values can be used to obtain an approximation for the first derivative of 3^x with respect to x. (3 marks)

h	0.01	0.001	0.0001	0.00001
$\dfrac{a^h - 1}{h}$				

(d) For what value of a does $\lim\limits_{h\to 0}\left(\dfrac{a^h - 1}{h}\right) = 1$? (1 mark)

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Question 17 **(9 marks)**

Spinners A and B are used in a game of chance, with equally likely outcomes of $2, 3, 4, 5, 6$ for spinner A and $2, 3, 4, 5$ for spinner B after each has been spun.

A player pays $\$2$ for one play of the game and will win $\$5$ if the outcomes of spinner A and spinner B are the same, $\$2$ if their outcomes differ by one, and nothing otherwise.

Let X be the profit (winnings minus payment) in dollars made by a player in one play of the game.

(a) Explain why X is a random variable and list all possible values it can take. (2 marks)

(b) Determine the expected value of X. (4 marks)

(c) Calculate the variance of X. (2 marks)

(d) Determine what the cost of one play of the game should be so that in the long run, a player will break even. (1 mark)

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Question 18 **(10 marks)**

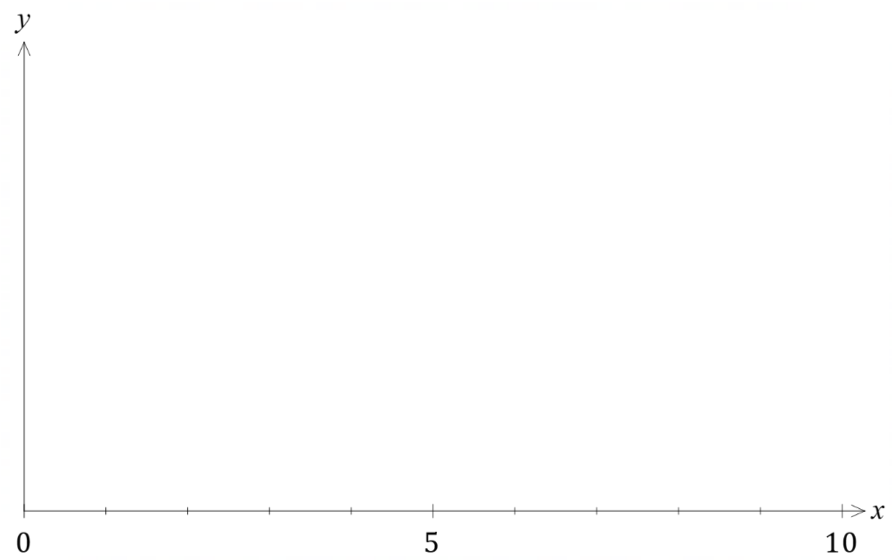
Consider the functions $f(x) = e^{0.125x}$ and $g(x) = mx$ for $x \geq 0$.

The positive constant m is such that the graphs of f and g always intersect.

Let R be the region enclosed by the y-axis and the graphs of f and g.

(a) Let $m = 0.5$.

(i) Sketch the graphs of f and g for $0 \leq x \leq 10$, showing the coordinates of the point where they intersect on the boundary of R. (3 marks)



(ii) Determine the area of R. (2 marks)

(b) Determine the maximum area of R. (5 marks)

Question 19 **(7 marks)**

The values of the polynomial functions f, g and h and some of their derivatives are shown in the table below.

x	$f(x)$	$g(x)$	$h(x)$	$f'(x)$	$g'(x)$	$h'(x)$
2	-1	-6	6	2	8	0
3	2	4	5	4	12	-2
4	7	18	2	6	16	-4

(a) Given that $h''(2) = -2$, describe the graph of $y = h(x)$ near $x = 2$. Justify your answer. (2 marks)

(b) Evaluate the derivative of $f(x) \cdot g(x)$ at $x = 2$. (2 marks)

(c) Evaluate the derivative of $\dfrac{f\big(g(x)\big)}{h(x)}$ at $x = 3$. (3 marks)

End of questions

Supplementary page

Question number: _________